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A Transformation of Solutions of Diffusion Equations Valid for Certain Initial and Boundary Conditions

Diffusivity problems, either with reference to a flux of material or a flux of heat, frequently arise in industrial, laboratory and biological systems. A full investigation of these must involve the formulation and solution of diffusion equations with the appropriate initial and boundary conditions. Unfortunately the mathematical problems involved are usually difficult, but this is counterbalanced by the fact that a large number of solutions of diffusion equations are present in the literature and in textbooks[1]. In the following, diffusion of a solute will be discussed, it being understood that only trivial modifications are required for the analogous conduction of heat problems.

Many mathematical solutions are available for those cases involving two adjacent regions occupied by two media, initially region *1* possessing a uniform concentration c_0' and region *2* possessing zero concentration of diffusing substance, diffusion subsequently occurring from region *1* to region *2*. A resistance to the flux may or may not be present at the interface. The mathematical solutions, giving the concentration of diffusing substance at any place and time in this type of system may readily be transformed to give the solutions for more difficult problems, where initially the diffusing substance has zero concentration in both regions and is then produced in region *1* at a constant rate m (e.g. mole/cm³/s), simultaneously diffusing into region *2*.

For the former systems the equations

$$\frac{\partial c_1'}{\partial t} = D_1 \nabla^2 c_1' \quad \text{with } c_1' = c_0' \text{ at } t = 0, \tag{1}$$

$$\frac{\partial c_2'}{\partial t} = D_2 \nabla^2 c_2' \quad \text{with } c_2' = 0 \text{ at } t = 0, \tag{2}$$

apply to regions *1* and *2* respectively.

Now consider the transformations

$$c_1 = \frac{m}{c_0'} \int_0^t c_1' \, dt \quad \text{and} \quad c_2 = \frac{m}{c_0'} \int_0^t c_2' \, dt .$$

Differentiating c_1 gives

$$\frac{\partial c_1}{\partial t} = \frac{m}{c_0'} \frac{\partial}{\partial t} \int_0^t c_1' \, dt = \frac{m \, c_1'}{c_0'} \tag{3}$$

and

$$D_1 \nabla^2 c_1 = \frac{m}{c_0'} D_1 \nabla^2 \int_0^t c_1' \, dt = \frac{m}{c_0'} \int_0^t D_1 \nabla^2 c_1' \, dt .$$

Substituting for $D_1 \nabla^2 c_1'$ from (1) results in

$$D_1 \nabla^2 c_1 = \frac{m}{c_0'} \int_0^t \frac{\partial c_1'}{\partial t} \, dt = \frac{m}{c_0'} (c_1' - c_0')$$

and substituting for mc_1'/c_0' from (3) gives

$$D_1 \nabla^2 c_1 = \frac{\partial c_1}{\partial t} - m .$$

Applying the same procedure to c_2 shows that

$$D_2 \nabla^2 c_2 = \frac{\partial c_2}{\partial t} .$$

Thus if c_1' and c_2' are solutions of (1) and (2) then c_1 and c_2 are solutions of

$$\frac{\partial c_1}{\partial t} = D_1 \nabla^2 c_1 + m$$

and

$$\frac{\partial c_2}{\partial t} = D_2 \nabla^2 c_2 .$$

These solutions will only apply for certain initial and boundary conditions.

As at zero time $c_1' = c_0'$ and $c_2' = 0$, then

$$c_1 = \frac{m}{c_0'} \int_0^t c_1' \, dt = 0$$

and similarly $c_2 = 0$. Thus c_1 and c_2 obey the conditions $c_1 = 0$ and $c_2 = 0$ at $t = 0$.

For the boundary conditions, consider first the interface between the two regions. If at this interface $c_1' = \alpha c_2'$ then it is easily seen that $c_1 = \alpha c_2$, where α is the partition coefficient of the diffusing substance between the two media. Also if $D_1 \, \partial c_1'/\partial n = D_2 \, \partial c_2'/\partial n$ at the interface, n being the normal to the boundary surface, then

$$D_1 \frac{\partial c_1}{\partial n} = \frac{m}{c_0'} \int_0^t D_1 \frac{\partial c_1'}{\partial n} \, dt = \frac{m}{c_0'} \int_0^t D_2 \frac{\partial c_2'}{\partial n} \, dt = D_2 \frac{\partial c_2}{\partial n} .$$

Thus c_1 and c_2 will satisfy $D_1 \, \partial c_1/\partial n = D_2 \, \partial c_2/\partial n$ at this boundary. Alternatively if the interfacial boundary condition satisfied by c_1' and c_2' is

$$D_1 \partial c_1'/\partial n = D_2 \partial c_2'/\partial n = P\,(\alpha \, c_2' - c_1') ,$$

corresponding to the existence of a membrane of permeability P at the interface, then $D_1 \, \partial c_1/\partial n = D_2 \, \partial c_2/\partial n = P\,(\alpha \, c_2 - c_1)$ forms the condition satisfied by c_1 and c_2, for

$$D_1 \frac{\partial c_1}{\partial n} = D_2 \frac{\partial c_2}{\partial n} = \frac{m}{c_0'} \int_0^t D_2 \frac{\partial c_2'}{\partial n} \, dt = \frac{m}{c_0'} \int_0^t P\,(\alpha \, c_2' - c_1') \, dt$$

$$= P \left[\alpha \frac{m}{c_0'} \int_0^t c_2' dt - \frac{m}{c_0'} \int_0^t c_1' \, dt \right] = P\,(\alpha \, c_2 - c_1) \cdot$$

[1] E.g. Carslaw and Jaeger, *Conduction of Heat in Solids* (Oxford Univ. Press, 1947).

It only remains to consider the conditions at the "outer" boundaries of the regions. The term "outer" boundary includes mathematical boundaries, e.g. at infinity, or the central point of a spherical region, or the axis of a cylindrical region. Let $\mathbf{r}$ and $\mathbf{r}_0$ be the radius vectors of general points within and on the "outer" boundaries of the system respectively. Then, either with reference to region *1* or region *2*, i.e. to c_1 or c_2, if c' is finite when $\mathbf{r} \to \mathbf{r}_0$, then

$$c = \frac{m}{c_0'} \int_0^t c' \mathrm{d}t$$

will also be finite as $\mathbf{r} \to \mathbf{r}_0$. Alternatively if $\partial c'/\partial n \to 0$ as $\mathbf{r} \to \mathbf{r}_0$ then

$$\frac{\partial c}{\partial n} = \frac{m}{c_0'} \int_0^t \frac{\partial c'}{\partial n} \mathrm{d}t \to 0 \quad \text{as} \quad \mathbf{r} \to \mathbf{r}_0,$$

n being the normal to the "outer" boundary.

Finally an illustration is given of the application of this transformation to linear diffusion in the normal direction across a plane interface. Consider the diffusing substance possessing the original concentration c_0' in region *1* ($0 < x < l$) and zero in region *2* ($x > l$) and then undergoing diffusion into region *2*. The differential equations

$$\frac{\partial c_1'}{\partial t} = D_1 \frac{\partial^2 c_0'}{\partial x^2} \quad \text{for} \quad 0 < x < l,$$
$$\frac{\partial c_2'}{\partial t} = D_2 \frac{\partial^2 c_2'}{\partial x^2} \quad \text{for} \quad x > l$$

apply, with initial conditions

$$c_1' = c_0',\ c_2' = 0 \quad \text{at} \quad t = 0,$$

and boundary conditions

$$\partial c_1'/\partial x = 0 \quad \text{at} \quad x = 0,$$
$$c_1' = \alpha c_2', \quad D_1 \partial c_1'/\partial x = D_2 \partial c_2'/\partial x \quad \text{at} \quad x = l,$$
$$c_2' \text{ finite as } x \to \infty.$$

The solutions are

$$c_1' = \frac{2}{\pi} c_0' \alpha \nu \int_0^\infty \frac{\exp(-D_1 t u^2) \cdot \sin lu \cdot \cos xu \cdot \mathrm{d}u}{u(\cos^2 lu + \alpha^2 \nu^2 \sin^2 lu)}$$

$$c_2' = \frac{2}{\pi} c_0' \nu \int_0^\infty \frac{\exp(-D_1 t u^2) \cdot \sin lu \cdot f(u) \cdot \mathrm{d}u}{u}$$

where

$$f(u) = \frac{[\cos lu \cdot \cos\{(x-l)\nu u\} - \alpha \nu \sin lu \cdot \sin\{(x-l)\nu u\}]}{(\cos^2 lu + \alpha^2 \nu^2 \sin^2 lu)}$$

and $\nu = (D_1/D_2)^{\frac{1}{2}}$.

Application of the transformations gives

$$c_1 = \frac{2 m \alpha \nu}{\pi D_1} \int_0^\infty \frac{\{1 - \exp(-D_1 t u^2)\} \cdot \sin lu \cdot \cos xu \cdot \mathrm{d}u}{u^3(\cos^2 lu + \alpha^2 \nu^2 \sin^2 lu)}$$

$$c_2 = \frac{2 m \nu}{\pi D_1} \int_0^\infty \frac{\{1 - \exp(-D_1 t u^2)\} \cdot \sin lu \cdot f(u) \cdot \mathrm{d}u}{u^3},$$

which form the solutions to the problem in which the diffusing substance is absent initially in both regions and is then produced uniformly throughout region *1* at rate m and diffuses into region *2*. The functions c_1 and c_2 form the solutions of the differential equations.

$$\frac{\partial c_1}{\partial t} = D_1 \frac{\partial^2 c_1}{\partial x^2} + m \quad \text{for} \quad 0 < x < l,$$
$$\frac{\partial c_2}{\partial t} = D_2 \frac{\partial^2 c_2}{\partial x^2} \quad \text{for} \quad x > l,$$

with initial conditions $c_1 = 0$, $c_2 = 0$ at $t = 0$, and boundary conditions for c_1 and c_2 which are identical with those for c_1' and c_2'.

D. G. O'SULLIVAN

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Zusammenfassung

Eine grosse Anzahl von Auflösungen für mathematische Gleichungen, die sich mit der Diffusion einer Substanz von einem Raum (Raum *1*), in dem sie ursprünglich in einheitlicher Konzentration vorhanden sind, in einen anderen Raum (Raum *2*) befassen, sind allgemein zugänglich. Es wird eine Methode beschrieben, die Auflösungen dieser Art in solche umwandelt, die sich unter der Bedingung verwenden lassen, dass die diffundierende Substanz ursprünglich abwesend ist, später jedoch mit konstanter Geschwindigkeit (in Raum *1*) produziert wird. Eine solche Methode der Umwandlung ist für eine Anzahl von Grenzschichtbedingungen (boundary conditions) anwendbar.

Isolation of Crystalline Aldosterone from the Urine of a Nephrotic Patient

Since 1950 one of the present authors (J.A.L.) and his collaborators have demonstrated the occurrence of a strongly sodium-retaining corticoid fraction in the urine of normal humans and especially of oedematous patients with nephrosis or congestive heart failure[1]. The active material, originally measured by bio-assay[2], was shown to move in ZAFFARONI's paper-chromatographic system with cortisone[3], behaving thus and in other respects[4] like the highly active mineralocorticoid from adrenal cortical extracts of SIMPSON and TAIT[5]. Similar activity in urine fractions of such patients has been observed by SINGER *et al.*[6], and recently by COPE *et al.*[7] who also

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[2] Q. B. DEMING and J. A. LUETSCHER, Jr., Proc. Soc. Exper. Biol. Med. *73*, 171 (1950). – J. A. LUETSCHER, Jr., and Q. B. DEMING, Transactions 2nd Conf. Renal Function, J. Macy Jr. Found., 155 (1951), New York. – B. B. JOHNSON, Endocrinol. *54*, 196 (1954).

[3] J. A. LUETSCHER, Jr., and B. B. JOHNSON, Amer. J. Med. *15*, 417 (1953); J. Clin. Invest. *32*, 585 (1953). – J. A. LUETSCHER, Jr., and B. J. AXELRAD, J. Clin. Endocrinol. *14*, 1086 (1954).

[4] J. A. LUETSCHER, Jr., and B. B. JOHNSON, J. Clin. Invest. *33*, 276 (1954).

[5] S. A. SIMPSON and J. F. TAIT, Endocrinol. *50*, 150 (1952) and later publications.

[6] B. SINGER and E. H. VENNING, Endocrinol. *52*, 623 (1953). – B. SINGER and J. WENER, Amer. Heart J. *45*, 795 (1953). – M. F. MCCALL and B. SINGER, J. Clin. Endocrinol. *13*, 1157 (1953).

[7] C. L. COPE and J. GARCIA, Brit. Med. J. *1954*, 1290.